# Group velocity of large-amplitude electromagnetic waves in a plasma

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The nonlinear group velocity of a laser pulse propagating in a cold underdense unmagnetized plasma is examined. Analytical expressions for the group velocity are derived for various pulse length regimes. These expressions reduce to the usual $\partial\omega/\partial k$ form for small amplitude and are verified for aribtrary amplitude using particle in cell simulations on a cyclic mesh. We find that the leading edge of a pulse moves at the linear group velocity and that the phase velocity of the excited wake is less than the group velocity of the pulse for symmetrically shaped pulses. The techniques used can be applied to other waves in a plasma.

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## I. INTRODUCTION

It has been recognized since the time of Rayleigh [1] that, besides the phase velocity $v_\phi$, there are several velocities associated with a wave which have physical significance. Rayleigh defined the group velocity to be the velocity of the envelope of a beat pattern constructed from two waves $(\omega_1,k_1)$ and $(\omega_2,k_2)$. This velocity is given by $(\omega_1-\omega_2)/(k_1-k_2)=\Delta\omega/\Delta k$, which reduces to the often quoted result $v_g=\partial\omega/\partial k$ in the limit that $\Delta\omega$ and $\Delta k$ approach zero. Other velocities associated with a wave are the energy transport velocity [1], signal velocity [1], and the packet velocity [1].

In particular, for a plane electromagnetic wave propagating in an unmagnetized plasma it is well known that the group velocity is given by

$$v_g=c\left[1-\frac{\omega_p^2}{\omega^2}\right]^{1/2}, \tag{1}$$

where $\omega_p$ is the plasma frequency and $\omega$ is the electromagnetic wave's frequency. In addition, the phase and group velocities are related by $v_g v_\phi=c^2$. Furthermore, it can be shown that both the energy transport velocity and the packet velocity are equal to $v_g$. However, these relationships are only true in the limit of infinitesimal wave amplitude. To understand the complications which arise for finite wave amplitude, consider Rayleigh's definition of the group velocity in terms of the beat velocity of two weakly nonlinear waves. Each wave now satisfies a generic dispersion relation which depends on the amplitude of both waves $\omega_1(k_1,k_2,A_1,A_2)$ and $\omega_2(k_1,k_2,A_1,A_2)$ where $A_1$ and $A_2$ are some measure of the wave amplitudes. Therefore, calculating $\Delta\omega/\Delta k$ of two waves or $\partial\omega/\partial k$ of one wave is now ambiguous because it depends on whether $A_1$, $A_2$, or some combination of both is kept fixed. This will be explicitly demonstrated in the next section. Furthermore, for a wave packet with a spectrum of frequencies, rather than a few discrete modes, it becomes impossible even to define a dispersion relation. As a result, there appears to be little in the literature concerning or even defining a nonlinear group velocity. An exception is the work of Lighthill [2] and Whitham [3] who considered systems with identifiable Lagrangians. They find that the only velocity with a well defined nonlinear counterpart is the energy transport velocity. In particular, Lighthill showed that the energy transport velocity is equal to $\partial\omega/\partial k$ when holding the average Lagrangian density $\langle\mathcal{L}\rangle$ divided by the frequency $\omega$ fixed, i.e., $v_g=(\partial\omega/\partial k)_{\langle\mathcal{L}\rangle/\omega}$. However, this method only gives the correct energy transport velocity if the Legendre transformation of $\mathcal{L}$ ( i.e., the Hamiltonian density) is equal to the physical energy density. For an arbitrary system this need not be the case and the velocity obtained may represent the flow of some quantity other than the energy.

Besides being of fundamental importance to nonlinear plasma physics, a nonlinear group velocity has practical implications for laser driven accelerator schemes. In either the plasma beat wave accelerator (PBWA) [4] or the laser wakefield accelerator (LWFA) [5] schemes the large transverse electric fields of high intensity lasers are converted into longitudinal electric fields of plasma waves. These plasma waves are used to accelerate particles to high energies. The longitudinal wave must have a phase velocity very close to the speed of light $c$ in order that accelerated particles and the wave do not dephase. It is straightforward to show [5] that the maximum energy gain $W^{\max}$ of an electron in a plasma wave of amplitude $\phi$ is given by

$$W^{\max}=4\epsilon\gamma_\phi^2 mc^2\ , \tag{2}$$

where $\gamma_\phi^2=1/(1-v_\phi^2/c^2)$, $v_\phi$ is the plasma wave phase velocity, and $\epsilon=e\phi/mc^2$. If one makes the ansatz that the wakefield phase velocity equals the laser group velocity then the maximum energy gain is a strong function of the group velocity. It is therefore useful to define $\gamma_g$ as

$$\gamma_g=\frac{1}{\sqrt{(1-v_g^2/c^2)}}\ . \tag{3}$$

It is worthwhile to point out that using the linear group velocity expression given by Eq. (1) gives

 

$$\gamma_g = \frac{\omega}{\omega_p} . \tag{4}$$

Akhiezer and Polovin [6] have shown that the nonlinear dispersion relation for purely transverse waves can be obtained from the linear dispersion relation by replacing $\omega_p^2$ with $\omega_p^2/\gamma_{\perp 0}$ where $\gamma_{\perp 0}^2 = 1 + (eE_0/m\omega c)^2/2$ and $E_0$ is the amplitude of the oscillating electric field. The nonlinear dispersion relation is simply $\omega_{\rm NL}^2 = \omega_p^2/\gamma_{\perp 0} + c^2k^2$. Physically this corresponds to a relativistic mass increase from the transverse quiver motion of the electron in the oscillating electromagnetic field. Therefore, it may appear that $\gamma_g$ for a nonlinear wave could be obtained by simply replacing $\omega_p^2$ with $\omega_p^2/\gamma_{\perp 0}$ in Eq. (4), which leads to $\gamma_g = (\omega/\omega_p)\gamma_{\perp 0}^{1/2}$. However, as we will show, simply deriving the group velocity by renormalizing the plasma frequency is not correct.

In this paper, we first explicitly show that the group velocity as defined by Rayleigh is ambiguous even in the weakly nonlinear limit. Next, as done in an earlier Letter [7], we derive an energy transport velocity valid for arbitrary amplitudes using the energy conservation equation. This expression is unambiguous and reduces to the well known small-amplitude expression. We then obtain a closed form expression for the group velocity in the long pulse limit. The energy conservation equation is then combined with the quasistatic equations to obtain a nonlinear group velocity for arbitrary pulse lengths. Recently, Chen and Sudan [8] obtained a Lagrangian density for the set of relativistic fluid Maxwell equations. We use this Lagrangian density and the method of Lighthill to recover the earlier expression for the group velocity. The analytical results are then verified using particle in cell (PIC) simulations on a cyclic mesh. We next examine the nonlinear group velocity obtained by using the quasistatic equations by themselves. We introduce a Lagrangian density of the quasistatic equations and use it to derive two conservation equations. We find that one of the conservation equations provides a transport velocity which is equal to the energy transport velocity obtained earlier. Lighthill's method is then applied to the quasistatic Lagrangian density and it is found that it gives an incorrect expression. The reason is that the Hamiltonian for this reduced system of equations does not represent the physical energy. Lastly, the consequences of this work to laser driven accelerator schemes are discussed.

We note that recently Kuehl *et al.* [9] made a weakly nonlinear analysis of the group velocity and wake excitation of short pulses. We present a fully nonlinear treatment. Furthermore, they concentrated on the times later than the pump depletion time, while we examine the early time behavior which is more relevant to the LWFA. In most LWFA schemes, Rayleigh diffraction and/or particle dephasing occurs sooner than pump depletion. However, if the pulse propagates long enough for pump depletion to occur then the frequency will decrease by photon deceleration [10], causing $v_g$ to decrease as given by the linear dispersion relation of Eq. (1).

## II. WEAKLY NONLINEAR REGIME

In this section we use the definition of Rayleigh to calculate the group velocity in the weakly nonlinear regime by evaluating the velocity of the beat pattern of two waves. The amplitude is expressed in terms of the normalized vector potential $a = eA/mc^2$, where $A$ is the vector potential. By weakly nonlinear we mean that only terms $O(a^2)$ are kept in the dispersion relations.

Rayleigh's recipe requires that the system admit a "beat" solution comprised of two distinct frequencies in order to evaluate $(\omega_1-\omega_1)/(k_1-k_2)$. The two waves $\omega_1, k_1$ and $\omega_2, k_2$ need not be solutions by themselves. In a general nonlinear system there is usually mixing between the various frequencies producing a spectrum of harmonics and sum frequencies rather than two distinct frequencies. However, in the limit of weak coupling there exist solutions dominated by just two frequencies because all of the waves at other frequencies are smaller by at least a factor of $O(a)$. In this limit coupled dispersion relations can be derived. The dispersion relation of a linearly polarized wave of frequency and wave number $\omega_1, k_1$ in the presence of another linearly polarized light wave of frequency and wave number $\omega_2, k_2$ is given by [11]

$$c^2k_1^2 = \omega_1^2 - \omega_p^2 \left[ 1 - \frac{1}{2}\left[\frac{3}{4} - \frac{\omega_1^2-\omega_p^2}{4\omega_1^2-\omega_p^2}\right]\frac{a_1^2}{c^2} - \frac{1}{4}\left[3 - \frac{(k_1+k_2)^2}{(\omega_1+\omega_2)^2-\omega_p^2}\right]\frac{a_2^2}{c^2}\right] , \tag{5}$$

where the amplitudes of the two light waves are expressed in terms of the normalized vector potentials $a_1 = eE_1/m\omega_1$ and $a_2 = eE_2/m\omega_2$. We note that for small amplitudes (nonrelativistic) $a_{1,2}$ is the transverse quiver velocity of an electron oscillating in the $E_{1,2}$ field.

Likewise, the dispersion relation of the second wave at $\omega_2, k_2$ is given by

$$c^2k_2^2 = \omega_2^2 - \omega_p^2 \left[ 1 - \frac{1}{2}\left[\frac{3}{4} - \frac{\omega_2^2-\omega_p^2}{4\omega_2^2-\omega_p^2}\right]\frac{a_2^2}{c^2} - \frac{1}{4}\left[3 - \frac{(k_1+k_2)^2}{(\omega_1+\omega_2)^2-\omega_p^2}\right]\frac{a_1^2}{c^2}\right] . \tag{6}$$

We now subtract Eq. (6) from Eq. (5) and use the relations

$$k_1^2 - k_2^2 = 2k\delta k + O(\delta k^2) , \tag{7}$$

$$\omega_1^2 - \omega_2^2 = 2\omega\delta\omega + O(\delta\omega^2) , \tag{8}$$

where $k \equiv (k_1+k_2)/2$, $\delta k \equiv k_1 - k_2$, $\omega \equiv (\omega_1+\omega_2)/2$, and $\delta\omega \equiv \omega_1 - \omega_2$. We assume that the waves are close to the same frequency and wave number so that we can neglect the $O(\delta k^2)$ and $O(\delta\omega^2)$ terms to obtain

$$\frac{2c^2k\delta k}{\omega_p^2} = \frac{2\omega\delta\omega}{\omega_p^2}\left[\frac{\beta}{4} - \frac{3}{8}\right]\left[\frac{a_1^2}{c^2} - \frac{a_2^2}{c^2}\right] - \frac{1}{2}\left[\alpha_1\frac{a_1^2}{c^2} - \alpha_2\frac{a_2^2}{c^2}\right] , \tag{9}$$

where

$$\alpha_{1,2} = \frac{\omega_{1,2}^2-\omega_p^2}{4\omega_{1,2}^2-\omega_p^2}$$

and

$$\beta = \frac{(k_1+k_2)^2}{(\omega_1+\omega_2)^2-\omega_p^2} .$$

We now use Eq. (9) to obtain the "beat" velocity of the two waves. This velocity is the group velocity of Rayleigh extended to weakly nonlinear waves and is defined by

$$v_g \equiv \frac{\delta\omega}{\delta k} . \tag{10}$$

However, in order to evaluate this velocity we must specify the relative amplitudes of the two waves. It is at this point that an ambiguity arises.

### A. Equal vector potentials

First we consider holding the vector potentials equal. Setting $a_1 = a_2 = v$ we obtain

$$2c^2 k\delta k = 2\omega\delta\omega + \frac{\omega_p^2}{2}(\alpha_1-\alpha_2)\frac{v^2}{c^2} . \tag{11}$$

The above definitions for $\delta\omega$ and $\delta k$ and some algebra lead to $\alpha_1-\alpha_2 = 6\omega\delta\omega/(4\omega^2-\omega_p^2)^2$. Substituting this into Eq. (11) results in

$$\frac{\delta\omega}{\delta k} = \frac{c^2 k}{\omega}\left[1+\frac{3}{2}\frac{v^2}{c^2}\frac{\omega_p^4}{(4\omega^2-\omega_p^2)^2}\right]^{-1} . \tag{12}$$

We see that the correction to the linear result $\delta\omega/\delta k = (k/\omega)c^2$ is of order $\omega_p^4/\omega^4$. We note that for circularly polarized light waves $\delta\omega/\delta k = (k/\omega)c^2$ exactly.

### B. Equal electric fields

Next we consider the beating of two waves of equal electric fields. The electric field is proportional to the frequency times the vector potential $[E = -(1/c)(\partial A/\partial t)]$. We therefore set $\omega_1 a_1 = \omega_2 a_2 = \omega v$. Equation (9) becomes

$$2kc^2\delta k = 2\omega\delta\omega\left[1+\frac{1}{2}\frac{v^2}{c^2}\frac{\omega_p^2}{\omega^2}-\frac{1}{4}\frac{v^2}{c^2}\frac{\omega_p^2}{\omega^2}\frac{4k^2}{4\omega^2-\omega_p^2} + O\left[\frac{\omega_p^4}{\omega^4}\right]\right] . \tag{13}$$

Neglecting $O(\omega_p^4/\omega^4)$ terms we obtain

$$\frac{\delta\omega}{\delta k} = \frac{c^2 k}{\omega}\left[1+\frac{1}{4}\frac{v^2}{c^2}\frac{\omega_p^2}{\omega^2}\right]^{-1} . \tag{14}$$

We note that Eq. (12) can be obtained by evaluating $\delta\omega/\delta k$ from the weakly nonlinear dispersion relation $\omega^2 = c^2k^2 + \omega_p^2(1-v^2/4c^2)$ holding $v$ fixed. Likewise Eq. (14) can be obtained in exactly the same way holding $\omega v$ fixed.

The most significant result is that Eqs. (12) and (14) are different to order $(\omega_p^2/\omega^2)(v^2/c^2)$. This demonstrates an ambiguity of using Rayleigh's definition for the group velocity. In the linear limit the velocity of the beats also equals the energy transport velocity. In the subsequent sections we show that the energy transport can be unambiguously determined.

## III. FULLY NONLINEAR FLUID MODEL

To derive an unambiguous expression for the nonlinear group velocity, we use the fully nonlinear fluid equations and Maxwell's equations, the only approximation being the neglect of kinetic effects. This is reasonable since we are dealing with underdense plasmas and $v_g \gg v_{\rm th}$ for any definition of $v_g$, where $V_{\rm th}$ is the thermal velocity. To obtain an energy transport velocity we must derive an energy continuity equation. Therefore we begin with Poynting's equation

$$\frac{\partial}{\partial t}\left[\frac{E^2+B^2}{8\pi}\right] + \nabla\cdot\frac{c}{4\pi}(\mathbf{E}\times\mathbf{B}) + \mathbf{J}\cdot\mathbf{E} = 0 \tag{15}$$

and the relativistic fluid momentum equation

$$\left[\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right]\mathbf{p} = -e\left[\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right] , \tag{16}$$

where $\mathbf{p} = m\gamma\mathbf{v}$.

To obtain an expression for $\mathbf{J}\cdot\mathbf{E}$, we substitute the curl of Eq. (16) into Faraday's law to obtain the relativistic vorticity equation

$$\frac{\partial}{\partial t}\left[\nabla\times\mathbf{p}-\frac{e}{c}\mathbf{B}\right] = \nabla\times\mathbf{v}\times\left[\nabla\times\mathbf{p}-\frac{e}{c}\mathbf{B}\right] . \tag{17}$$

Therefore, if $\nabla\times\mathbf{p}-(e/c)\mathbf{B}=0$ initially, then it is true for all times. This can be rewritten as $\nabla\times\mathbf{P}=0$ where we have used $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{P}$ is the canonical momentum, $\mathbf{P}\equiv\mathbf{p}-(e/c)\mathbf{A}$. In other words, if the canonical momentum is initially irrotational then it remains so. Substituting $\mathbf{B}=(c/e)\nabla\times\mathbf{p}$ into Eq. (16) and using $p^2 = m^2c^2(\gamma^2-1)$ along with some vector identifies, we can rewrite the relativistic fluid momentum equation in a more useful form as

$$\frac{\partial}{\partial t}\mathbf{p} = -e\mathbf{E} + \nabla(mc^2\gamma) \tag{18}$$

and since

$$\mathbf{J} = -ne\mathbf{v} \tag{19}$$

we can write

$$\mathbf{J}\cdot\mathbf{E} = \frac{\partial}{\partial t}(nmc^2\gamma) + \nabla\cdot(nmc^2\gamma\mathbf{v}) . \tag{20}$$

Substituting Eq. (20) into Poynting's equation gives

$$\frac{\partial}{\partial t}\left[\frac{E^2+B^2}{8\pi}+nmc^2\gamma\right] + \nabla\cdot\left[\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}+nmc^2\gamma\mathbf{v}\right] = 0 . \tag{21}$$

We want the leftmost term of Eq. (21) to be the time derivative of the total energy. To achieve this we subtract $mc^2$ times the continuity equation

$$mc^2\left[\frac{\partial n}{\partial t}+\nabla\cdot(n\mathbf{v})\right]=0 \tag{22}$$

from Eq. (21) to obtain the conservation of energy equation

$$\frac{\partial}{\partial t}\left[\frac{E^2+B^2}{8\pi}+nmc^2(\gamma-1)\right]+\nabla\cdot\left[\frac{c}{4\pi}\mathbf{E}\times\mathbf{B}+nmc^2(\gamma-1)\mathbf{v}\right]=0\ . \tag{23}$$

A local energy transport velocity can be found by noting the above expression is of the form

$$\frac{\partial}{\partial t}(U)+\nabla\cdot(\mathbf{S})=0\ , \tag{24}$$

where $U$ is an energy density and $\mathbf{S}$ is an energy flux. We average Eq. (24) over the high frequency oscillations ($\langle\ \rangle$) and define a local energy transport velocity as

$$\mathbf{v}_E\equiv\frac{\langle\mathbf{S}\rangle}{\langle U\rangle}\ . \tag{25}$$

We use this definition, rather than $\mathbf{v}_E\equiv\langle\mathbf{S}/U\rangle$, because it is also the time rate of change of the average position of a finite length pulse. If we define the position of a finite length pulse as the energy weighted expectation value

$$\bar{x}=\frac{\int dx\, xU}{\int dx\ U}\ , \tag{26}$$

where the integration is over the length of the pulse, then the velocity of the pulse, i.e., the group velocity, is given by

$$\bar{v}_g\equiv\frac{d}{dt}\bar{x}=\frac{\int dx\ x\,\partial U/\partial t}{\int dx\ U}=\frac{\int dx\ S}{\int dx\ U}=\frac{\bar{\mathbf{S}}}{\bar{U}}\ , \tag{27}$$

where the conservation of $U$, $(d/dt)\int dx\ U=0$, is implicitly used. Therefore, in what follows we refer to the local energy transport velocity as the local group velocity $\mathbf{v}_g$.

Using these definitions, we find from Eq. (23) that

$$\mathbf{v}_g=\frac{\langle(c/4\pi)\mathbf{E}\times\mathbf{B}+nmc^2(\gamma-1)\mathbf{v}\rangle}{\langle(E^2+B^2)/8\pi+nmc^2(\gamma-1)\rangle}\ . \tag{28}$$

Equation (28) is a general expression for the velocity of energy transport. This technique can also be used for any wave in a plasma once a solution for the fields is found. In addition, it could be used to analyze any problem involving energy transport in more than one direction such as relativistic self-focusing. In this paper we are concerned with the group velocity of electromagnetic waves propagating in unmagnetized plasmas. It turns out that the solutions for electromagnetic waves in plasmas depend upon the pulse length as well as polarization. Therefore we evaluate Eq. (28) for different polarizations and pulse lengths separately.

### A. Long pulse limit

To determine an analytic expression for the group velocity using Eq. (28) we require analytical solutions for the fields $\mathbf{E}$, $\mathbf{B}$, $\gamma$, and $\mathbf{v}$. Akhiezer and Polovin [6] provide such solutions for waves of the form $f(x-v_\phi t)$. In the limit of electromagnetic pulses much longer than $2\pi c/\omega_p$, the longitudinal electric field can be neglected, and from Faraday's law $E_\perp=(v^\phi/c)B_\perp$. We note that for linear polarization there is a longitudinal electric field $E_\parallel$ associated with the second harmonic of the transverse field; however, for $\omega_p^2/\omega^2\ll1$ it is much smaller than $E_\perp$. Using the relation that $E_\perp=(v_\phi/c)B_\perp$ we reduce Eq. (28) to

$$v_g=(2c^2/v_\phi)\times\frac{1+(\omega_p^2/\omega^2)(\langle(\gamma-1)nv_x\rangle/n_0)(v_\phi/\langle p_\perp^2\rangle)}{1+c^2/v_\phi^2+2(\omega_p^2/\omega^2)[\langle(\gamma-1)n/n_0\rangle/\langle p_\perp^2\rangle]}\ , \tag{29}$$

where $v_g=\mathbf{v}_g\cdot\hat{\mathbf{x}}$, $p_\perp=eE_\perp/m\omega$.

Expressions for $v_\phi$ can be found in Ref. [6] for both linearly and circularly polarized light. The expressions can be summarized as

$$v_\phi^2=\frac{1}{1-\omega_p^2/\omega^2\gamma_{\perp0}}\ , \tag{30}$$

where $\gamma_{\perp0}^2\equiv1+\langle p_\perp^2/m^2c^2\rangle$. If we define a nonlinear parameter $p_0\equiv eE_0/mc\omega$, where $E_0$ is the amplitude of electromagnetic wave $E_\perp$, then $\langle p_\perp^2/m^2c^2\rangle=p_0^2$ is used for circularly polarized light and $\langle p_\perp^2/c^2\rangle=p_0^2/2$ is used for linearly polarized light. The circular polarization expression is exact while the linear polarization expression is valid for $\omega_p^2/\omega^2\ll1$ [6,12,13]. We treat circularly polarized light and linearly polarized light separately.

#### 1. Circularly polarized light

For circularly polarized waves there is no density perturbation so $n=n_0$. In addition, the electron motion is purely transverse so $v_x=0$ and $\langle\gamma\rangle=\gamma_{\perp0}$. Using the result of Akhiezer and Polovin [6] we find

$$1+\frac{c^2}{v_\phi^2}=2-\frac{\omega_p^2}{\omega^2\gamma_{\perp0}}\ . \tag{31}$$

Substituting this into Eq. (29) gives

$$v_g=\frac{c^2/v_\phi}{1+(\omega_p^2/2\omega^2)[(\gamma_{\perp0}-1)/p_0^2-1/2\gamma_{\perp0}]}\ , \tag{32}$$

which reduces to

$$v_g=\frac{c^2/v_\phi}{1+(\omega_p^2/2\omega^2)[(\gamma_{\perp0}-1)/\gamma_{\perp0}(\gamma_{\perp0}+1)]}\ , \tag{33}$$

where $\gamma_{\perp0}$ and $v_\phi$ are defined above. We note that Eq. (33) is exact for circularly polarized light and that no assumptions about $\omega_p^2/\omega^2$ are made. Before addressing the case of linear polarization, we expand Eq. (33) to order $\omega_p^2/\omega^2$ to obtain

$$v_g \simeq 1 - \frac{1}{\gamma_{\perp 0}+1}\frac{\omega_p^2}{\omega^2} .$$

### 2. *Linearly polarized light*

For linearly polarized light the motion is not purely transverse so $v_x \neq 0$ and there are density perturbations driven at the second harmonic, so $n \neq n_0$. However, Akhiezer and Polovin [6] have shown that for sufficiently underdense plasmas the dispersion relation Eq. (31) is still valid for linearly polarized light. We start by noting that if we have solutions of the form $n = n(x - v_\phi t)$ and $v_x = v_x(x - v_\phi t)$ then $\partial/\partial t = -v_\phi \partial/\partial x$ and the continuity equation gives

$$\frac{n v_x}{v_\phi} = \delta n \equiv n - n_0 . \tag{34}$$

Inserting $n v_x = \delta n v_\phi$ and Eq. (31) into Eq. (29) we obtain

$$v_g = (c^2/v_\phi)\frac{1+2(\omega_p^2/\omega^2)[\langle(\gamma-1)\delta n v_\phi^2\rangle/c^2 n_0 p_0^2]}{1+(\omega_p^2/2\omega^2)[(\gamma_{\perp 0}-1)/\gamma_{\perp 0}(\gamma_{\perp 0}+1)]+2(\omega_p^2/\omega^2)[\langle(\gamma-1)\delta n\rangle/n_0 p_0^2]} . \tag{35}$$

We note that the only difference between Eq. (35) and the circular polarization result given by Eq. (33) is the

$$2\frac{\omega_p^2}{\omega^2}\frac{\langle(\gamma-1)\delta n\rangle v_\phi^2}{n_0 p_0^2}$$

term in the numerator and the

$$2\frac{\omega_p^2}{\omega^2}\frac{\langle(\gamma-1)\delta n\rangle}{n_0 p_0^2}$$

in the denominator. Since $v_\phi^2 - c^2 \sim \omega_p^2/\omega^2$ these two terms are equivalent to $O(\omega_p^4/\omega^4)$. Also, since $p_0^2 \sim \gamma_{\perp 0}^2 - 1$ and $\delta n/n_0 \sim (\gamma_{\perp 0}^2-1)/4\gamma_{\perp 0}^2$ [13] these terms are always small for underdense plasmas. We can cancel these terms by noting that Eq. (35) is of the form $(1+a)/(1+a+b)$ which reduces to

$$(1+a)(1-a-b) \simeq 1 - b + O(a^2, b^2)$$

for $|a|, |b| \ll 1$. Since Eq. (33) is of the form $1/(1+b) \simeq 1-b$, Eqs. (35) and (33) are equal to $O(\omega_p^2/\omega_0^2)$.

Therefore for sufficiently underdense plasmas the nonlinear group velocity is the same for both circularly polarized and linearly polarized light, and is given by Eq. (33). For small amplitudes, i.e., $\gamma_\perp \sim 1$, we recover the usual relations $v_\phi v_g = c^2$ and $v_g = \partial\omega/\partial k = c\sqrt{1-\omega_p^2/\omega^2}$. However, for nonlinear amplitudes this simple relation between $v_\phi$ and $v_g$ does not hold and Eq. (33) is not recovered by simply differentiating the nonlinear dispersion relation while holding $\gamma$ fixed. Furthermore, it does not reduce to either Eq. (12) or Eq. (14) in the weakly relativistic limit.

The importance of the nonlinear corrections to the group velocity can be most easily demonstrated by examining $\gamma_g$. Substituting Eq. (33) into Eq. (3) gives

$$\gamma_g \sim \left[\frac{\gamma_{\perp 0}+1}{2}\right]^{1/2} \omega/\omega_p \tag{36}$$

in the $\omega_p \ll \omega$ limit. We see that this is not the same as replacing $\omega_p^2$ with $\omega_p^2/\gamma_{\perp 0}$ in Eq. (4). Since the energy gain in the LWFA is proportional to $\gamma_g^2$ this could lead to a factor of 2 reduction in the electron energy.

Before examining the short pulse regime, we address the validity of the long pulse limit. The assumption of wavelike solutions of the form $f(x - v_\phi t)$ only requires that the pulse length $l_0$ be larger than $c/\omega$. This is reasonable even for short pulses in sufficiently underdense plasmas. The crucial assumption is the neglect of wakefield effects. The magnitude of the ponderomotively excited wakefield $E_\parallel$ is given by $eE_\parallel/m\omega_p c = \pi p_0^2 c^2/\omega_p^2 l_0^2$ [14]. The long pulse limit is valid when $E_\parallel \ll E_\perp$. This leads to a condition on the pulse length $l_0\omega_p/c \gg (\omega_p/\omega)(\pi p_0)^{1/2}$.

### B. Short pulse limit

For pulses which are less than a few plasma wavelengths $2\pi c/\omega_p$, a plasma wakefield is excited and the relation $E = (v_\phi/c)B$ does not hold. However, if we denote the laser field with $E_\perp$ and the plasma wakefield with $E_\parallel$, then $E_\perp = (v_\phi/c)B_\perp$ and $B_\parallel = 0$. There are several differences from the long pulse limit. First, $E_\parallel$ is no longer small and contributes to the electric field energy density in the denominator of Eq. (28). Second, $\delta n$ and $V_x$ are associated with the wake and not the harmonics, and therefore they cannot be neglected even for circularly polarized light. Third, $\gamma$ now contains longitudinal motion so $\langle\gamma\rangle \neq \gamma_{\perp 0}$.

When a wake is formed by a short pulse Eq. (34) is not valid because $n$ is of the form $n(x - v_w t)$, where $v_w$ is the phase velocity of the wake not the light wave. However, $c^2 - v_w^2 \sim \omega_p^2/\omega^2$, so to $O(\omega_p^2/\omega^2)$ we may write $n v_x/c = \delta n$. Using this relationship, $E_\perp = (v_\phi/c)B_\perp$ and keeping $E_\parallel$ in the electric field energy density, Eq. (28) becomes

$$v_g = (c^2/v_\phi)\frac{1+(\omega_p^2/\omega^2)[\langle(\gamma-1)\delta n\rangle/n_0 p_0^2]}{\frac{1}{2}(1+c^2/v_\phi^2)+(\omega_p^2/\omega^2)(\langle\gamma-1\rangle/p_0^2)+(e^2/2\omega^2c^2)(E_\parallel/p_0^2)+(\omega_p^2/\omega^2)[\langle(\gamma-1)\delta n\rangle/n_0 p_0^2]} , \tag{37}$$

where we have assumed circular polarization so $\langle p_\perp^2 \rangle = p_0^2$. As in the case of long pulse linearly polarized light, there is a term in the denominator which is identical to the rightmost term in the numerator. We may cancel these terms if

$$\frac{\omega_p^2}{\omega^2}\frac{\langle(\gamma-1)\delta n\rangle}{n_0 p_0^2} \ll 1 .$$

As before $p_0^2=\gamma_{\perp 0}^2-1$; however, $\delta n/n_0$ can be large because of wakefield excitation. It has been shown [15] that $\max(\delta n/n_0)\sim\gamma_{\perp 0}^4/4$. Therefore, the condition on neglecting these terms is $p_0 \ll (\omega_p/\omega)^{2/3}$. Under these approximations, we can write Eq. (37) as

$$v_g = \frac{c^2/v_\phi}{\frac{1}{2}(1+c^2/v_\phi^2)+(\omega_p^2/\omega^2)(\langle\gamma-1\rangle/p_0^2)+(e^2/2m^2\omega^2c^2)(\langle E_\parallel^2\rangle/p_0^2)} . \tag{38}$$

We are in the short pulse regime so we make use of the quasistatic approximation [12] to reduce Eq. (38). The quasistatic approximation consists of neglecting $\partial/\partial\tau$ in the continuity equation and in the longitudinal equation of motion after a mathematical transformation has been made from the $(x,t)$ to the $(\xi=x-ct, \tau=t)$ coordinates. The physical interpretation of this approximation is that the laser pulse does not evolve (appears static) during a pulse duration. The result is a coupled set of nonlinear equations for the scalar potential $\Phi$ and the vector potential $\mathbf{A}$. In a later section we calculate the group velocity that the quasistatic equations give by themselves. The quasistatic equations for the normalized potentials $\phi=e\Phi/mc^2$ and $\mathbf{a}=e\mathbf{A}/mc^2$ are

$$\left[\frac{2}{c}\frac{\partial^2}{\partial\xi\partial\tau}-\frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\right]\mathbf{a}=k_p^2\frac{\mathbf{a}}{1+\phi} , \tag{39}$$

$$\frac{\partial^2}{\partial\xi^2}\phi=\frac{1}{2}k_p^2\left[\frac{1+a^2}{(1+\phi)^2}-1\right] , \tag{40}$$

where $k_p=\omega_p/c$.

The quasistatic approximation gives a nonlinear dispersion relation of $c^2/v_\phi^2=1-\omega_p^2/\omega^2\chi$, where $\chi=1+\phi$ and the normalized potentials are $\phi\equiv e\Phi/mc$ and $\mathbf{a}\equiv e\mathbf{A}/mc^2$. Since $E_\parallel=-\nabla\Phi$ we can write $eE_\parallel/m\omega_p c=-k_p\partial\chi/\partial\xi$. The main advantage to using the quasistatic approximation is that it gives a relation between $\gamma$ and $\chi$. The quasistatic approximation gives

$$1+\phi=\gamma(1-v_x/c)=\gamma-p_\parallel/mc .$$

This expression and the relationship $\gamma^2=1+a^2+p_\parallel^2/m^2c^2$ give

$$\gamma=\frac{\gamma_\perp^2+\chi^2}{2\chi} , \tag{41}$$

where $\gamma_\perp^2=1+a^2$. We substitute $c^2/v_\phi^2=1-\omega_p^2/\omega^2\chi$, $eE_\parallel/m\omega_p c=-k_p\partial\chi/\partial\xi$, and Eq. (41) into Eq. (38) to obtain a local value for the group velocity of a light pulse in the $\omega_p^2/\omega^2\ll 1$ limit,

$$v_g=\frac{c^2/v_\phi}{1+(\omega_p^2/2\omega^2)\{(1/p_0^2)[(k_p^2\partial\chi/\partial\xi)^2+(\gamma_\perp^2+\chi^2-2\chi)/\chi]-1/\chi\}} . \tag{42}$$

The functional dependence of $\chi$ is described by

$$\frac{\partial^2\chi}{\partial\xi^2}=\frac{1}{2}k_p^2\left[\frac{\gamma_\perp^2}{\chi^2}-1\right] , \tag{43}$$

where $k_p=\omega_p/c$. For short pulses $\chi$ is predominantly a plasma wave wake. This wake is the basis of the LWFA. The long pulse limit an be recovered by neglecting $\partial\chi/\partial\xi$ in Eqs. (42) and (43). For arbitrary length pulses it is difficult to obtain a closed form expression for the local $v_g$ becuase $\chi$ must first be solved using Eq. (43) and then substituted into Eq. (42).

### C. Ultrashort limit

We now examine the group velocity for an ultrashort pulse or, equivalently, for the leading edge of a long pulse. It has been shown [12] that $\phi$ grows from the front of the pulse on $\omega_p^{-1}$ time scales. To see this we assume $\phi\ll 1$ and Eq. (43) becomes

$$\frac{\partial^2\chi}{\partial\xi^2}=\frac{\partial^2\phi}{\partial\xi^2}=\tfrac{1}{2}k_p^2(\gamma_{\perp 0}^2-1)=\tfrac{1}{2}k_p^2p_0^2 . \tag{44}$$

We integrate this equation with $\phi=\partial\phi/\partial\xi=0$ at the front of the pulse ($\xi=0$), to obtain

$$\phi=\left[p_0\frac{k_p\xi}{2}\right]^2 . \tag{45}$$

For simplicity we have assumed a constant amplitude pulse, i.e., a square pulse, and set $a(\xi)=p_0$. A square pulse will give the largest wakefield; therefore we can assume $\phi\ll 1$ when $\xi<2c/p_0\omega_p$ for any pulse shape.

Using Eq. (45) we find that

$$\left[\frac{\partial\chi}{\partial\xi}\right]^2=\left[\frac{p_0^2k_p^2\xi}{2}\right]^2=p_0^2k_p^2\phi .$$

The local group velocity of the very front of a pulse, or the group velocity of ultrashort pulses $l_0\ll c/\omega_p$, can be obtained by letting $(\partial\chi/\partial\xi)^2=k_p^2\phi$ and $\phi\ll 1$ in Eq. (42).

By Taylor expanding in $\phi$, we find that $1/x=1-\phi+O(\phi^2)$ and

$$\frac{\gamma_\perp^2+\chi^2-2\chi}{p_0^2\chi}=\frac{p_0^2+1+(1+\phi)^2-2(1+\phi)}{(1+\phi)p_0^2}=(1-\phi)+O(\phi^2)\ . \tag{46}$$

Substituting this into Eq. (42) and neglecting terms of order $\phi^2$ we obtain

$$v_g=\frac{c^2/v_\phi}{1+(\omega_p^2/2\omega^2)[\phi+(1-\phi)-(1-\phi)]}=\frac{c^2/v_\phi}{1+(\omega_p^2/2\omega^2)\phi}\ . \tag{47}$$

Since

$$\frac{1}{v_\phi}=1-\frac{\omega_p^2}{2\omega^2}(1-\phi)+O(\phi^2)$$

we find that

$$v_g=1-\frac{\omega_p^2}{2\omega^2}\ . \tag{48}$$

We therefore conclude that the group velocity for the leading edge of long pulses, as well as the group velocity for ultrashort pulses, is the linear group velocity irrespective of the wave amplitude. A similar result has been shown [12] for the phase velocity of the leading edge of a pulse. It is this fact which prevents the relativistic guiding of the leading $c/\omega_p$ of a pulse.

### D. Method of Lighthill

Recently, Chen and Sudan [8] obtained a Lagrangian density for the set of relativistic fluid Maxwell equations. They found that the appropriate Lagrangian density is given by

$$\mathcal{L}=\frac{1}{2}\left[(\nabla\times\mathbf{a})^2-\left[\nabla\phi+\frac{\partial\mathbf{a}}{\partial t}\right]^2\right]+\omega_p^2 n\left[\frac{\partial\psi}{\partial t}+\gamma-\phi-1\right]+\omega_p^2\phi\ , \tag{49}$$

where $\nabla\psi=\mathbf{p}-\mathbf{a}$.

We use this Lagrangian density and the method of Lighthill to obtain an expression for the group velocity in the long pulse limit. For simplicity we assume circularly polarized light. Using the solutions for circularly polarized light given earlier we find that the average Lagrangian density is given by

$$\langle\mathcal{L}\rangle=\frac{p_0^2}{2}(k^2-\omega^2)+\omega_p^2\phi\ . \tag{50}$$

Using the relationships $\gamma_{\perp 0}^2=1+p_0^2$ and $1+\phi=\gamma_{\perp 0}$ we obtain

$$\frac{\langle\mathcal{L}\rangle}{\omega}=\frac{\gamma_{\perp 0}^2-1}{2}(k^2/\omega-\omega)+\omega_p^2\frac{\gamma_{\perp 0}-1}{\omega}\ . \tag{51}$$

Lighthill's method gives $v_g=(\partial\omega/\partial k)_{\langle\mathcal{L}\rangle/\omega}$. Using the fact that

$$\left[\frac{\partial\omega}{\partial k}\right]_{\langle\mathcal{L}\rangle/\omega}=-\frac{\partial f}{\partial k}\Big/\frac{\partial f}{\partial\omega}\ ,$$

where $f=\langle\mathcal{L}\rangle/\omega$, we find that

$$v_g=\frac{2c^2k/\omega}{1+c^2k^2/\omega^2+2\omega_p^2/\omega^2(\gamma_{\perp 0}-1)}\ . \tag{52}$$

Using Eq. (31) we recover Eq. (33), where $c$ and $\omega_p$ were reincorporated.

## IV. SIMULATIONS

These analytical expressions were investigated using the electromagnetic particle in cell code ISIS which has recently been modified to include a cyclic mesh. The cyclic mesh is a technique for the following short pulses by removing columns of cells from far behind the pulse and placing (cycling) them at the front of the pulse with fresh particles. The cycling rate is chosen so that the grid moves at the speed of light $c$. This cyclic mesh code has been benchmarked with other well established PIC codes.

These simulations are done in the $x$-$y$ plane with linear polarization in the $z$ direction. We initialize a laser pulse in vacuum and let it propagate in the $x$ direction into the plasma. The initial profile is of the form

$$\frac{eE_\perp}{m\omega_p c}=\frac{p_0}{c}\frac{\omega}{\omega_p}\sin^2\left[\frac{\pi}{2}\frac{x}{l_0}\right]\sin\left[\frac{\omega_0}{c}x\right]\ . \tag{53}$$

We first verified the expressions for the phase velocity for both circularly and linearly polarized light. This was done by tracking the position of wave crests. The scaling with both $\gamma$ and $\omega/\omega_p$ was confirmed. The results for linear polarization are presented in Fig. 1 where we plot the phase velocity $v_\phi$ versus $p_0$ for $\omega/\omega_p=5.0$ (solid circles) and 10.0 (open circles). In addition, we plot the theoretical phase velocity given by Eq. (30) for the respective frequencies.

The group velocity was measured by calculating the pulse's energy weighted expectation position defined by $\bar{x}=\int dx\, xE_\perp^2/\int dx\, E_\perp^2$ at every time step and then evaluating $v_g=(d/dt)\bar{x}$ at the end of the simulation. We weighted the position using $E_\perp^2$ rather than the entire en-

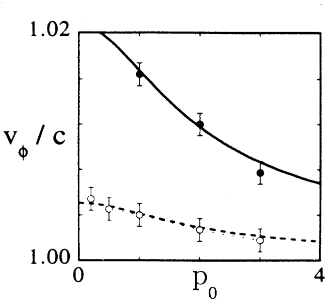


FIG. 1. Phase velocity $v_\phi$ versus wave amplitude $p_0$ for $\omega/\omega_p=5$ (solid circles) and 10 (open circles) from simulations. Solid and dashed lines are from Eq. (30) for the respective frequencies.

ergy $U$. It can be shown that this gives small errors to $v_g$ in the long pulse limit. To show this we note that $U$ can be written as

$$U=\frac{1}{8\pi}\left[1+\frac{c^2}{v_\phi^2}\right]E_\perp^2+nmc^2(\gamma-1)\ . \tag{54}$$

Using the relations $1+c^2/v_\phi^2=2-\omega_p^2/\omega^2\gamma_{\perp 0}$, $p_0\equiv eE_\perp/mc\omega$, and $\gamma=1+p_0^2$, we can write

$$U=\frac{E_\perp^2}{4\pi}\left[1+\frac{\omega_p^2}{\omega^2}f(\gamma)\right]\ , \tag{55}$$

where $f(\gamma)=(\gamma-1)/2\gamma(\gamma+1)$. We note that $f(\gamma)$ vanishes for $\gamma=1$ and $\gamma\gg 1$ with a maximum value of 0.08 at $\gamma=2$. Using this expression for $U$ it is easy to show that

$$\frac{\int dx\, xU}{\int dx\, U}=\frac{\int dx\, xE_\perp^2}{\int dx\, E_\perp^2}[1+(f_1-f_2)O(\omega_p^2/\omega^2)+f_1f_2O(\omega_p^4/\omega^4)]\ , \tag{56}$$

where $f_1=\int dx\, f(\gamma)xE_\perp^2/\int dx\, xE_\perp^2$ and $f_2=\int dx\, f(\gamma)E_\perp^2/\int dx\, E_\perp^2$. Numerical calculations show that $f_1-f_2\sim 10^{-7}$; therefore weighting the position using $E_\perp^2$ rather than $U$ is reasonable.

If this is not done in the short pulse limit the calculated group velocity would be artificially lower than the theoretical value because the wake left behind the pulse, being a space charge wave, has zero energy flux but a nonzero energy density. A typical simulation result is shown in Fig. 2 for $\omega/\omega_p=5.0$, $p_0=3$, and $l_0=10c/\omega_p$ where we plot $\bar{x}$ versus time. We see that in the vacuum region the curve is flat, indicating a group velocity very close to the speed of light. Thus the numerical dispersion associated with the field solver is much less than the plasma dispersion. Numerical dispersion can be controlled [16] by reducing the grid size $\delta x$ and keeping the ratio of the grid size to the time step $\delta t$ close to unity. Inside the plasma the group velocity (the slope of Fig. 2) remains very constant for many plasma periods, thus allowing for very accurate measurements. As the pulse propagates further into the plasma, pump depletion occurs. Pump depletion results from pulse distortion and a lowering of the pulse's frequency which leads to a reduction in $v_g$. However, from a simple energy conservation analysis it can be shown that the pump depletion time scales as $\omega^2/\omega_p^2$ so this does not effect the measurements done earlier on in time.

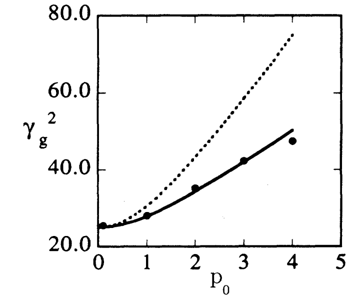


FIG. 3. $\gamma_g^2=1/(1-v_g^2/c^2)$ versus wave amplitude $p_0$ for $\omega/\omega_p=5$. Data points are for simulations with pulse length of $140c/\omega_p$ and solid line is theory given by Eq. (33). Dotted line is the result of replacing $\omega_p^2$ with $\omega_p^2/\gamma_{\perp 0}$ in the linear theory.

In Fig. 3 we plot $\gamma_g^2$, defined by Eq. (3), versus $p_0$ for $\omega/\omega_p=5.0$. We chose to plot $\gamma_g^2$ because of its sensitivity to $v_g$. Any discrepancies in $v_g$ resulting from different definitions will be enhanced. The solid dots are the result of using values of $v_g$ from the PIC simulations and the solid line is the result of using Eq. (33) for $v_g$. The simulations were done for numerous values of $p_0$ for a long pulse with a Gaussian rise and fall of $l_0=20c/\omega_p$ and a flat section of $100c/\omega_p$. The laser frequency was chosen to be $\omega/\omega_p=5.0$ in order to lessen the computer time. From Fig. 3 we see excellent agreement between the theoretical expression Eq. (33) and the PIC result. This agreement for the long pulse group velocity demonstrates that weighting the position with $E_\perp^2$ rather than $U$ is accurate. In addition, we plot $(\omega/\omega_p)\gamma_{\perp 0}^{1/2}$ (dashed line) in Fig. 3. This clearly shows that replacing $\omega_p^2$ with $\omega_p^2/\gamma_{\perp 0}$ in the linear group velocity given by Eq. (4) is incorrect. Simulations using higher values of $\omega/\omega_p$ were also carried out, and the scaling of Eq. (33) with frequency was verified. The results are shown in Fig. 4 where we plot $v_g$ versus $\omega$ for $p_0=2$ and Eq. (33). Here we also see very good agreement between the theoretical expression Eq.

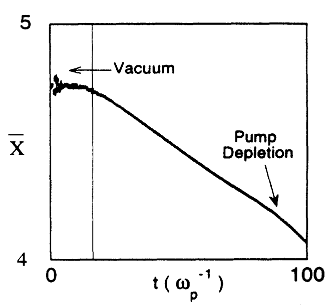


FIG. 2. Weighted expectation position versus time from computer simulation. The slope of this curve is the group velocity.

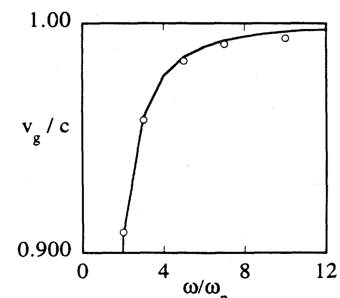


FIG. 4. Group velocity $v_g$ versus frequency $\omega/\omega_p$ for $p_0=2.0$. Data points are from simulation and solid line is from theory.

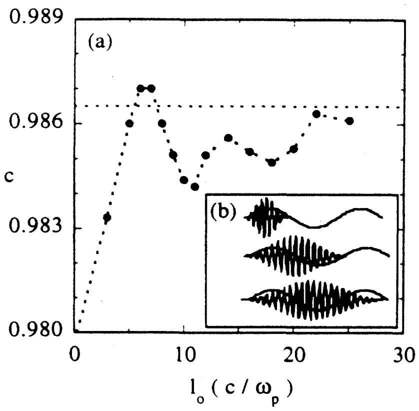


FIG. 5. (a) Group velocity $v_g$ versus pulse width for $p_0=3.0$ and $\omega/\omega_p=5$. Dashed line is an $E_1^2$ weighted average of Eq. (33) over the pulse shape. (b) Sketch showing the relative positions of the laser pulse to density depression (wake) for three pulse lengths.

(33) and the PIC result.

Simulations of short pulses were also carried out to verify the implication of Eq. (42). The results are summarized in Fig. 5(a) where we plot the group velocity versus the pulse width for $p_0=3.0$ and $\omega/\omega_p=5.0$. Based on Eq. (42) we expect the group velocity of a short pulse, i.e., $l_0 \leq c/\omega_p$, to approach the linear value. This is consistent with Fig. 5(a) where the dashed line through the leftmost points approaches the linear $v_g$. No simulation points are available for smaller values of $l_0$ because there are too few cycles within the pulse to define a single frequency.

The group velocity begins to increase as the pulse width increases. This occurs because $v_\phi$ decreases and the dominant term in Eq. (42) is the numerator. Physically, the increase in $v_g$ is due to the reduction of the local value of $\omega_p$ caused by the density depression of the wake and the relativistic mass increase. The relative positions of the laser pulse to the density depression are illustrated in Fig. 5(b) where we give a sketch for three different pulse lengths. We see the ultrashort pulse resides entirely in the first density compression. This density compression exactly cancels the relativistic mass increase from the quiver velocity. This is the physical reason why ultrashort pulses move at the linear group velocity (and have the linear phase velocity) regardless of amplitude. As the pulse length increases the pulse samples regions of density depression and the group velocity increases. For longer pulse lengths, regions of the pulse will again reside in subsequent density compressions. This leads to a modulation in the group velocity with pulse length. This scenario is seen in Fig. 5(a) where the group velocity oscillates as a function of $l_0$. The periodicity corresponds to the wake's wavelength which is a function of this amplitude, and hence a function of $p_0$. Similar curves were obtained for other values of $p_0$.

As the pulse width increases further the group velocity asymptotes to the long pulse expression. The amplitude of the oscillation decreases because the wake's amplitude decreases with pulse length. The asymptotic limit is not that given in Eq. (33) because Gaussian shaped pulses were used in the simulations of Fig. 5(a). We therefore calculated an $E_1^2$ weighted average of Eq. (33) over the pulse shape and this value is plotted as the horizontal dashed line. The agreement between the calculated value and the rightmost simulation points is excellent.

A crucial issue for the laser wakefield accelerator is the dephasing between the particles and the wake. Dephasing is when the particle accelerates so much that it begins to outrun the wave. As the particle starts to run up the moving potential hill it gets decelerated. To avoid dephasing, $v_w$ should be as close to $c$ as possible. Previously, it has always been assumed that $v_w=v_g$. However, this relation can be altered by pulse shaping, linear or nonlinear dispersion, photon acceleration or deceleration, and pulse distortion. We have carried out simulations to investigate the nonlinear dependence between $v_g$ and $v_w$. The wake's phase velocity was determined by tracking the first minimum of $E_\parallel$. Sample results are presented in Fig. 6 where $v_w$ (solid circles) and $v_g$ (triangles) are plotted versus $p_0$ for $\omega/\omega_p=5.0$ and $l_0=6c/\omega_p$.

We find that $v_w=v_g$ only for linear values of $p_0$ and symmetric pulses. However, as shown in Fig. 6, as $p_0$ increases, $v_g$ increases while $v_w$ decreases. This opposite dependence on $p_0$ is not paradoxical because the part of the pulse which is generating the wake need not travel at the average velocity of the pulse. This point at the front of the pulse which generates the wake gradually etches backward due to local pump depletion [17]. As a result, wakes excited by the leading edge of the laser should cause the excitation point to etch backward while wakes excited by the trailing edge of the laser should cause the excitation point to etch forward. We therefore expect $v_w > v_g$ for pulses with a long rise time and sharp fall, and $v_w < v_g$ for pulses with a sharp rise and a long fall. Indeed this is what is observed in simulations of such pulses as shown in Fig. 6. Data points labeled with the solid squares are for simulations done with a pulse with a sharp rise ($l_0=4c/\omega_p$) and a long fall ($l_0=10c/\omega_p$), whereas data points labeled with the open squares are for simulations done with a pulse with a long rise

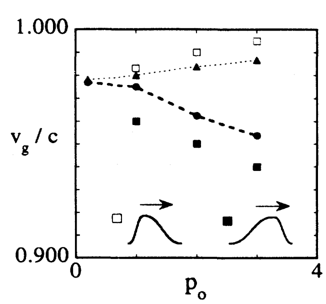


FIG. 6. Wakefield phase velocity $v_w$ (triangle) and pulse group velocity $v_g$ (circle) versus amplitude $p_0$ for symmetrically shaped pulses and $v_w$ (squares) for asymmetrically shaped pulses.

($l_0=10c/\omega_p$) and a sharp fall ($l_0=14c/\omega_p$). Clearly, the pulse with the sharp fall excites a wake with the fastest phase velocity; it is even faster than that of a symmetric pulse. Thus tailoring the shape of the laser pulse could serve as a technique for optimizing energy gain in the LWFA.

## V. QUASISTATIC APPROXIMATION

Recently, much work has been done on nonlinear laser plasma interactions using the quasistatic equations [12,14]. In some of these works the emphasis is on phenomena which involve group velocity concepts such as the wakefield phase velocity in the LWFA and relativistic self-focusing. Therefore in this section we use the quasistatic equations to obtain a nonlinear group velocity in terms of an energy transport velocity. The quasistatic equations are given in the preceding section as Eqs. (39) and (40).

In order to obtain an energy transport velocity we need to derive an energy conservation equation. It is well known that Lagrangian systems admit energy-momentum conservation equations. We find the following Lagrangian density function which generates Eqs. (39) and (40):

$$\mathcal{L}(\mathbf{a},\mathbf{a}_\xi,\mathbf{a}_\tau,\phi,\phi_\xi)=\tfrac{1}{2}\phi_\xi^2+\frac{1}{2c^2}a_\tau^2-\frac{1}{c}\mathbf{a}_\xi\cdot\mathbf{a}_\tau-\frac{k_p^2}{2}\left[\frac{1+a^2}{1+\phi}+\phi-1\right], \tag{57}$$

where the subscripts $\xi$ and $\tau$ denote $\partial/\partial\xi$ and $\partial/\partial\tau$, respectively. Verifying that this is the Lagrangian density is easily done by checking that

$$\frac{\partial\mathcal{L}}{\partial a}-\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial a_\tau}-\frac{\partial}{\partial\xi}\frac{\partial\mathcal{L}}{\partial a_\xi}=0$$

and

$$\frac{\partial\mathcal{L}}{\partial\phi}-\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\phi_\tau}-\frac{\partial}{\partial\xi}\frac{\partial\mathcal{L}}{\partial\phi_\xi}=0$$

recover Eq. (39) and Eq. (40).

Having a Lagrangian density function, we use the stress-energy tensor notation of Goldstein [18] and obtain the following energy conservation equations:

$$\frac{\partial T_{00}}{\partial\tau}+\frac{\partial T_{01}}{\partial\xi}=0 \tag{58}$$

and

$$\frac{\partial T_{10}}{\partial\tau}+\frac{\partial T_{11}}{\partial\xi}=0\ , \tag{59}$$

where the components of the stress-energy tensor $T$ are defined by

$$T_{00}\equiv\frac{\partial\mathcal{L}}{\partial\phi_\tau}\phi_\tau+\frac{\partial\mathcal{L}}{\partial\mathbf{a}_\tau}\mathbf{a}_\tau-\mathcal{L}\ , \tag{60}$$

$$T_{01}\equiv\frac{\partial\mathcal{L}}{\partial\phi_\xi}\phi_\tau+\frac{\partial\mathcal{L}}{\partial\mathbf{a}_\xi}\mathbf{a}_\tau\ , \tag{61}$$

$$T_{10}\equiv\frac{\partial\mathcal{L}}{\partial\phi_\tau}\phi_\xi+\frac{\partial\mathcal{L}}{\partial\mathbf{a}_\tau}\mathbf{a}_\xi\ , \tag{62}$$

$$T_{11}\equiv\frac{\partial\mathcal{L}}{\partial\phi_\xi}\phi_\xi+\frac{\partial\mathcal{L}}{\partial\mathbf{a}_\xi}\mathbf{a}_\xi-\mathcal{L}\ . \tag{63}$$

The $T_{00}$ component is the Hamiltonian for the system.

Since a Lagrangian which is a function of the two variables $(\psi,\tau)$ admits two conservation equations, we can define two different velocities for a wave. For mechanical systems $T_{00}$ is the energy density (Hamiltonian), so we define the energy transport velocity from Eq. (58) as

$$v_{00}=c+\frac{\langle T_{01}\rangle}{\langle T_{00}\rangle}\ . \tag{64}$$

However, we can also define the transport velocity from Eq. (59) as

$$v_{10}=c+\frac{\langle T_{11}\rangle}{\langle T_{10}\rangle}\ , \tag{65}$$

where we have used $\partial/\partial t=\partial/\partial\tau-c\partial/\partial\xi$ to express $v_g$ in the $(x,t)$ coordinates and $\langle\ \rangle$ represents averaging over the fast oscillations.

However, for this Lagrangian density of the reduced set of equations we must explicitly determine what $T_{00}$ or $T_{10}$ represents before identifying either Eq. (64) or Eq. (64) as the pulse's group velocity. Using the above expressions for $T_{ij}$ and Eq. (57) we explicitly find the components of $T_{ij}$ to be

$$T_{00}=\frac{1}{2c^2}a_\tau^2-\tfrac{1}{2}\phi_\xi^2+\frac{k_p^2}{2}\left[\frac{1+a^2}{1+\phi}+\phi-1\right], \tag{66}$$

$$T_{01}=\frac{\partial}{\partial\xi}\left[\tfrac{1}{2}\phi_\xi\phi_\tau-\frac{1}{2c^2}a_\tau^2\right], \tag{67}$$

$$T_{10}=\frac{1}{c^2}\mathbf{a}_\tau\cdot\mathbf{a}_\xi-\frac{1}{c}a_\xi^2\ , \tag{68}$$

$$T_{11}=\tfrac{1}{2}\phi_\xi^2-\frac{1}{2c^2}a_\tau^2+\frac{k_p^2}{2}\left[\frac{1+a^2}{1+\phi}+\phi-1\right]. \tag{69}$$

From Eq. (68), we see that $T_{10}$ is proportional to the transverse electromagnetic energy density. To verify this, we consider the normalized perpendicular energy density defined by $\mathcal{E}_\perp\equiv[(E_\perp^2+B_\perp^2)/8\pi]/n_0mc^2$. Using the relations $\mathbf{E}_\perp=-(1/c)(\partial\mathbf{A}/\partial t)$ and $\mathbf{B}_\perp=\partial\mathbf{A}/\partial z$ along with $\partial/\partial t=\partial/\partial\tau-c\partial/\partial\xi$ and $\partial/\partial z=\partial/\partial\xi$ we find that

$$\mathcal{E}_\perp=\frac{c^2}{2\omega_p^2}\left[\frac{1}{c^2}a_\tau^2-\frac{2}{c}\mathbf{a}_\tau\cdot\mathbf{a}_\xi+2a_\xi^2\right]. \tag{70}$$

In the quasi-static approximation it is assumed that $\partial\mathbf{A}/\partial\tau\ll\partial\mathbf{A}/\partial\xi$, so $T_{10}=-\mathcal{E}_\perp$. Therefore we use Eq. (65) to calculate the group velocity.

As with the preceding section, we evaluate Eq. (65) in the long pulse limit. In this limit the pulse length is longer than many plasma wavelengths, so no wake is excited and $\phi_\xi=\phi_\tau=0$. Setting $\partial^2\chi/\partial\xi^2=0$ in Eq. (39) gives $(1+\chi)^2=1+\langle a^2\rangle\equiv\gamma_{\perp 0}^2$ where we have neglected the harmonics. Assuming that

$$a(\tau,\xi)=\frac{a_0(\tau)}{2}\exp(ik\xi)+\text{c.c.}\ ,$$

Eq. (39) gives

$$\frac{\partial a}{\partial \tau}=\frac{-ick_p^2}{2k}\frac{a_0}{\gamma_{\perp 0}}\exp(ik\xi)+\text{c.c.} \tag{71}$$

and

$$\frac{\partial a}{\partial \xi}=ika_0\exp(ik\xi)+\text{c.c.} \tag{72}$$

The time averaged component $\langle T_{10}\rangle$ is

$$\langle T_{10}\rangle=-\frac{1}{c}\left[k^2+\frac{k_p^2}{2\gamma_{\perp 0}}\right]a_0^2 \tag{73}$$

and $\langle T_{11}\rangle$ becomes

$$\langle T_{11}\rangle=k_p^2(\gamma_{\perp 0}-1)-\left[\frac{k_p^2}{2k}\right]^2\frac{a_0^2}{2\gamma_{10}^2}\ . \tag{74}$$

Using Eq. (65) we find

$$v_g=1-\frac{1}{(\gamma_{\perp 0}+1)}\frac{\omega_p^2}{\omega^2} \tag{75}$$

to order $\omega_p^2/\omega^2$. We note that this is identical to expanding Eq. (33) to order $\omega_p^2/\omega^2$. This is a significant result for two reasons. It demonstrates that the transverse field energy $\mathcal{E}_\perp$ moves at the same velocity as the total energy $U$ to order $\omega_p^2/\omega^2$. This validates weighting the position of the pulse with $\mathcal{E}_\perp$ rather than with $U$ when calculating $v_g$ in the simulations. Additionally, contrary to our remarks in Ref. [7], the quasistatic equations describe the nonlinear group velocity correctly to order $\omega_p^2/\omega^2$.

It is illustrative to consider the velocity of Eq. (64) instead of Eq. (65). In this case the energy density of the system is $\langle T_{00}\rangle=k_p^2(\gamma_{\perp 0}-1)$ and its transport velocity is

$$v_g=c\left[1-\frac{\gamma_{\perp 0}+1}{\gamma_{10}^2}\left[\frac{k_p}{2k}\right]^2\right]\ .$$

This is the quasistatic result we alluded to in Ref. [7]. The reason this does not agree with Eq. (75) is that this represents the velocity at which $\gamma_{\perp 0}$ moves and not the velocity at which $\mathcal{E}_\perp$, i.e., $\langle T_{00}\rangle$, moves (energy transport velocity). Furthermore, this is also the velocity obtained from the method of Lighthill. To verify this we note that, in the speed of light variables, Lighthill's method gives $v_g=1+(\partial\Omega/\partial k)_{\langle\mathcal{L}\rangle/\Omega}$, where $\Omega=\omega-k$. We must use $\Omega$ rather than $\omega$ because we consider a plane wave of the form

$$a(\tau,\xi)=\frac{a_0}{2}\exp(ikx-i\omega t)+\text{c.c.}$$

which in the speed of light variables $(a_0/2)\exp(ik\xi-i\Omega\tau)+\text{c.c.}$ The averaged Lagrangian density is given by $\langle\mathcal{L}\rangle=-k\Omega a_0^2/2-(\gamma_{\perp 0}-1)$. Using the fact that

$$\left[\frac{\partial\Omega}{\partial k}\right]_{\langle\mathcal{L}\rangle/\Omega}=-\frac{\partial f}{\partial k}\Big/\frac{\partial f}{\partial\Omega}\ ,$$

where $f=\langle\mathcal{L}\rangle/\Omega$ along with the dispersion relation $\Omega=1/k\gamma_{\perp 0}$ we find that

$$v_g=\left[1-\frac{\gamma_{\perp 0}+1}{\gamma_{10}^2}\left[\frac{k_p}{2k}\right]^2\right]\ .$$

The method of Lighthill implicitly assumes that $T_{00}$ is the physical energy. However, for a Lagrangian that represents some arbitrary system, $T_{00}$ may not represent the quantity of interest.

## VI. CONCLUSIONS

In summary, we have investigated several issues concerning the concept of a nonlinear group velocity. First, the difficulties with defining a nonlinear velocity of a pulse using the conventional approach of group velocity are shown in the weakly nonlinear limit. Second, we introduced a more general definition in terms of energy transport and calculated an analytical expression from the fully nonlinear fluid equations. In addition, we showed that the energy transport velocity of an ultrashort pulse or, equivalently, the velocity of the leading edge of a long pulse is that of the linear group velocity. Third, we used PIC simulations to verify these expressions and examine finite pulse length effects. Last, we examined the phase velocity of the wakefield in regimes relevant to the LWFA. We find that the wake's phase velocity $v_w$ is not simply give by the driving pulse's group velocity. The relation between $v_w$ and $v_g$ depends on the pulse shape. In particular, $v_w < v_g$ for symmetric pulses.

To illustrate the importance of this decrease in $v_w$ for symmetric pulses in possible near term experiments, we simulated $\omega/\omega_p=20$ and $p_0=2.0$. This corresponds to a 35 fs, 1 $\mu$m laser pulse with $I=5\times10^{18}$ W/cm$^2$ propagating through a plasma of $n=4\times10^{19}$ cm$^{-3}$. We find that $\gamma_\omega^2=230$ while the analytic $\gamma_g^2=550$. Therefore the maximum energy gain is half of what is naively expected. However, if we use a pulse with a slow rise time and a fast fall we find $\gamma_g^2=600$.

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